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On the parity of exponents in the standard factorization of $n!$ [☆]

Yong-Gao Chen

Department of Mathematics, Nanjing Normal University, Jiangsu Province, Nanjing 210097, China

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Abstract

Let p_1, p_2, \dots be the sequence of all primes in ascending order. The following result is proved: for any given positive integer k and any given $\varepsilon_i \in \{0, 1\}$ ($i = 1, 2, \dots, k$), there exist infinitely many positive integers n with

$$e_1(n!) \equiv \varepsilon_1 \pmod{2}, e_2(n!) \equiv \varepsilon_2 \pmod{2}, \dots, e_k(n!) \equiv \varepsilon_k \pmod{2},$$

where $e_i(n!)$ denotes the exponent of the prime p_i in the standard factorization of positive integer $n!$. In 1997 Berend proved a conjecture of Erdős and Graham, that is, the conclusion with all $\varepsilon_i = 0$.

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1. Introduction

Let p_1, p_2, \dots be the sequence of all primes in ascending order. For a positive integer n , let $e_i(n)$ be the nonnegative integer with $p_i^{e_i(n)} \mid n$ and $p_i^{e_i(n)+1} \nmid n$. In 1997, Berend [1] proved a conjecture of Erdős and Graham (cf. [3, p. 77]) by showing that

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E-mail address: ygchen@pine.njnu.edu.cn.

for every positive integer k there exist infinitely many positive integers n with

$$e_1(n!) \equiv 0 \pmod{2}, e_2(n!) \equiv 0 \pmod{2}, \dots, e_k(n!) \equiv 0 \pmod{2}.$$

It is clear that $n = 1$ is a solution. The initial value $n = 1$ is very useful in Berend's proof. For any other pattern, we do not know if an initial value exists. An interesting generalization is (see [2]).

Problem. Given a positive integer k and $\varepsilon_1, \dots, \varepsilon_k \in \{0, 1\}$, does there exist some $n > 1$ with

$$e_i(n!) \equiv \varepsilon_i \pmod{2}, \quad i = 1, 2, \dots, k?$$

For $2 \leq k \leq 5$ Chen and Zhu has verified that every pattern of length k appears, and believes that for each positive integer k , every pattern of length k appears (see [2, p. 2]). Chen and Zhu [2] showed that if there is an initial value n , then the initial value is bounded by an explicit bound depending on k and there are infinitely many such n with the difference of adjacent n less than an explicit bound depending on k . Recently, Sander [4] posed the following conjecture:

Conjecture. Let q_1, q_2, \dots, q_k be distinct primes, and let $\varepsilon_i \in \{0, 1\}$ ($i = 1, 2, \dots, k$). Then there are infinitely many positive integers n such that

$$e'_{q_i}(n!) \equiv \varepsilon_i \pmod{2},$$

where $e'_{q_i}(n!)$ is the exponent of q_i in the standard factorization of $n!$.

The conjecture is equivalent to a similar conjecture with the assumption that q_1, q_2, \dots, q_k are the first k primes if we do not fix k . Sander [4] proved the conjecture for $k = 2$. In the present paper, we improve the method in Sander [4] and show that for any pattern there exists an initial value n . This implies that the answer to the above problem is affirmative and the above conjecture is true for all k .

Theorem 1. For any given positive integer k and any $\varepsilon_i \in \{0, 1\}$ ($i = 1, 2, \dots, k$), there exist infinitely many positive integers n with

$$e_1(n!) \equiv \varepsilon_1 \pmod{2}, e_2(n!) \equiv \varepsilon_2 \pmod{2}, \dots, e_k(n!) \equiv \varepsilon_k \pmod{2}.$$

2. Proof

Lemma 1 (Sander [4]). Let n be a positive integer with p_i -adic expansion $n = n_s p_i^s + \dots + n_1 p_i + n_0$, $0 \leq n_j < p_i$ ($j = 0, 1, \dots, s$). Then

$$e_i(n!) \equiv \begin{cases} \sum_{j \geq 1} n_j \pmod{2} & \text{for } i = 1, \\ \sum_{2 \nmid j} n_j \pmod{2} & \text{for } i > 1. \end{cases}$$

Lemma 2. *If there exist k integers n_1, n_2, \dots, n_k and 2^k integers m_1, m_2, \dots, m_{2^k} with $n_i + m_j > 0$ ($i = 1, 2, \dots, k$; $j = 1, 2, \dots, 2^k$) and*

$$(e_1((n_1 + m_j)!), e_2((n_2 + m_j)!), \dots, e_k((n_k + m_j)!)), \quad j = 1, 2, \dots, 2^k$$

represent all parities modulo 2, then there exist infinitely many positive integers n for each of which all $n + m_j > 0$ and

$$(e_1((n + m_j)!), e_2((n + m_j)!), \dots, e_k((n + m_j)!)), \quad j = 1, 2, \dots, 2^k$$

represent all parities modulo 2.

Proof. Assume that $2^{2^t} > \max_{i,j} \{n_i + m_j\}$. By the Chinese Remainder Theorem, there exist infinitely many positive integers $n > \max_i n_i$ such that

$$n \equiv n_i \pmod{p_i^{2^t+1}}, \quad i = 1, 2, \dots, k.$$

Noting that

$$n + m_j = \frac{n - n_i}{p_i^{2^t}} p_i^{2^t} + n_i + m_j, \quad 0 < n_i + m_j < p_i^{2^t},$$

by Lemma 1, we have

$$e_i((n + m_j)!) \equiv e_i\left(\left(\frac{n - n_i}{p_i^{2^t}}\right)!\right) + e((n_i + m_j)!) \pmod{2}.$$

Thus, for $j = 1, 2, \dots, 2^k$, we have

$$\begin{aligned} & (e_1((n + m_j)!), e_2((n + m_j)!), \dots, e_k((n + m_j)!)) \\ & \equiv \left(e_1\left(\left(\frac{n - n_1}{p_1^{2^t}}\right)!\right), e_2\left(\left(\frac{n - n_2}{p_2^{2^t}}\right)!\right), \dots, e_k\left(\left(\frac{n - n_k}{p_k^{2^t}}\right)!\right) \right) \\ & \quad + (e_1((n_1 + m_j)!), e_2((n_2 + m_j)!), \dots, e_k((n_k + m_j)!)) \pmod{2}. \end{aligned}$$

Lemma 2 follows from the above congruent equality by the observation that for any given $(a_1, a_2, \dots, a_k) \in \mathbf{Z}^k$,

$$(a_1, a_2, \dots, a_k) + (x_1, x_2, \dots, x_k)$$

goes through all 2^k parities modulo 2 as (x_1, x_2, \dots, x_k) goes through all 2^k parities modulo 2. This completes the proof of Lemma 2. \square

Theorem 2. For any given $k \geq 1$, there exist 2^k positive integers l_1, l_2, \dots, l_{2^k} such that

$$(e_1(l_j!), e_2(l_j!), \dots, e_k(l_j!)), \quad j = 1, 2, \dots, 2^k$$

represent all parities modulo 2.

Proof. We use induction on k . For $k = 1$, Theorem 2 is trivial by taking $l_1 = 1$ and $l_2 = 2$. Suppose that Theorem 2 is true for k . That is, there exist 2^k positive integers l_1, l_2, \dots, l_{2^k} such that

$$(e_1(l_j!), e_2(l_j!), \dots, e_k(l_j!)), \quad j = 1, 2, \dots, 2^k$$

represent all parities modulo 2. Let t be an even integer with

$$2^t > \max_j l_j,$$

and let

$$n_1 = 2^{t+1} + 2^t,$$

$$n_i = p_i^t \quad (2 \leq i \leq k), \quad n_{k+1} = p_{k+1}^{t+1},$$

$$m_j = \begin{cases} -1 - l_j & \text{for } 1 \leq j \leq 2^k, \\ l_{j-2^k} & \text{for } 2^k + 1 \leq j \leq 2^{k+1}. \end{cases}$$

Then l_j has the p_i -adic expansion

$$a_{jit}p_i^t + a_{ji(t-1)}p_i^{t-1} + \dots + a_{ji1}p_i + a_{ji0}$$

with $a_{jit} = 0$ and $0 \leq a_{jiv} \leq p_i - 1$ for all j, i, v . For $1 \leq j \leq 2^k$, we have

$$n_1 + m_j = 2^{t+1} + (1 - a_{j1(t-1)})2^{t-1} + \dots + (1 - a_{j11})2 + 1 - a_{j10},$$

$$n_1 + m_{2^k+j} = 2^{t+1} + 2^t + a_{j1(t-1)}2^{t-1} + \dots + a_{j11}2 + a_{j10},$$

$$n_i + m_j = (p_i - 1 - a_{ji(t-1)})p_i^{t-1} + \dots + (p_i - 1 - a_{ji1})p_i + p_i - 1 - a_{ji0},$$

$$n_i + m_{2^k+j} = p_i^t + a_{ji(t-1)}p_i^{t-1} + \dots + a_{ji1}p_i + a_{ji0}, \quad 2 \leq i \leq k,$$

$$n_{k+1} + m_j = (p_{k+1} - 1 - a_{j(k+1)t})p_{k+1}^t + \cdots + (p_{k+1} - 1 - a_{j(k+1)1})p_{k+1} + p_{k+1} - 1 - a_{j(k+1)0},$$

$$n_{k+1} + m_{2^k+j} = p_{k+1}^{t+1} + a_{j(k+1)t}p_{k+1}^t + \cdots + a_{j(k+1)1}p_{k+1} + a_{j(k+1)0}.$$

By Lemma 1 and $2|t$, for $1 \leq i \leq k$, noting that $p - 1 - a \equiv a \pmod{2}$ for prime $p > 2$, we have

$$e_i((n_i + m_j)!) \equiv e_i((l_j)!) \pmod{2},$$

$$e_i((n_i + m_{2^k+j})!) \equiv e_i((l_j)!) \pmod{2},$$

$$e_{k+1}((n_{k+1} + m_j)!) \equiv e_{k+1}((l_j)!) \pmod{2}, \quad (1)$$

$$e_{k+1}((n_{k+1} + m_{2^k+j})!) \equiv 1 + e_{k+1}((l_j)!) \pmod{2}. \quad (2)$$

Let $\varepsilon_i \in \{0, 1\}$ ($i = 1, 2, \dots, k+1$). By the induction hypothesis we may take j , $1 \leq j \leq 2^k$, such that

$$e_i((l_j)!) \equiv \varepsilon_i \pmod{2}, \quad i = 1, 2, \dots, k.$$

For the j , by (1) and (2), we may choose $u = j$ or $2^k + j$ such that

$$e_{k+1}((n_{k+1} + m_u)!) \equiv \varepsilon_{k+1} \pmod{2}.$$

Thus

$$e_i((n_i + m_u)!) \equiv \varepsilon_i \pmod{2}, \quad i = 1, 2, \dots, k+1.$$

This means that

$$(e_1((n_1 + m_j)!), e_2((n_2 + m_j)!), \dots, e_k((n_k + m_j)!)), \quad j = 1, 2, \dots, 2^{k+1}$$

represent all parities modulo 2. By Lemma 2 there exists a positive integer n with all $n + m_j > 0$ and

$$(e_1((n + m_j)!), e_2((n + m_j)!), \dots, e_k((n + m_j)!)), \quad j = 1, 2, \dots, 2^{k+1}$$

represent all parities modulo 2. This completes the proof of Theorem 2. \square

Theorem 1 follows from Theorem 2 and Lemma 2.

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